JOURNAL OF RESEARCH of the National Bureau of Standards — B. Mathematics and Mathematical Physics
Vol. 70B, No. 3, July–September 1966

Finding a Rank-Maximizing Matrix Block

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(May 11, 1966)

An algorithm is developed for the following problem: Given three matrices of respective dimensions $s\times s$, $s\times t$, and $t\times s$, to find a $t\times t$ matrix such that the $(s+t)\times(s+t)$ matrix formed from the four blocks has maximum rank.

Key words: Matrix, block, rank, algorithm.

Let A, B, C be matrices (over any field) of respective dimensions $s\times s$, $s\times t$, $t\times s$. We shall show how to find a $t\times t$ matrix X such that

$$M(X)=\begin{bmatrix} A & B \\ C & X \end{bmatrix}$$

has maximum possible rank $\rho(M(X))$, or equivalently has minimum possible deficiency

$$\delta(M(X))=s+t-\rho(M(X)).$$

Let β' and γ' be the respective ranks of the matrices

$$[A \quad B], \begin{bmatrix} A \\ C \end{bmatrix}.$$

It is readily seen that for any choice of X,

$$\delta(M(X))\geqslant \max\,(s-\beta',\, s-\gamma'). \tag{1}$$

We will show that X can be chosen so that *equality* holds in (1). Thus

$$\min_X \delta(M(X))=\max\,(s-\beta',\, s-\gamma'). \tag{2}$$

For the special case $\beta'=\gamma'=s$, this was proved by Pearl.[1]

Denote the rank of A by α. Find nonsingular $s\times s$ matrices U and V which bring A into Smith normal form, i.e.,

$$UAV=I_\alpha\dot{+}0_{s-\alpha}.$$

[1] Martin Pearl, On Normal *EPr* Matrices, Mich. Math. J. **8**, 33–37 (1961), (Theorem 2).

$M(X)$ will have the same rank and deficiency as

$$M_1(X)=\begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix} M(X) \begin{bmatrix} V & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} UAV & UB \\ CV & X \end{bmatrix},$$

which we repartition as

$$M_1(X)=\begin{bmatrix} I_\alpha & 0 & B_1 \\ 0 & 0_{s-\alpha} & B_2 \\ C_1 & C_2 & X \end{bmatrix}$$

where $\qquad [C_1C_2]=CV, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}=UB.$

$M_1(X)$, in turn, has the same rank and deficiency as

$$M_2(X)=\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -C_1 & 0 & I \end{bmatrix} M_1(X) \begin{bmatrix} I & 0 & -B_1 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

$$=\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & B_2 \\ 0 & C_2 & X-C_1B_1 \end{bmatrix}.$$

Let β and γ be the respective ranks of B_2 and C_2. Since $[A\ B]$ has the same rank as

$$U[A\ B](V\dot{+}I_t)\begin{bmatrix} I & 0 & -B_1 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} I_\alpha & 0 & 0 \\ 0 & 0 & B_2 \end{bmatrix}$$

we see that $\beta'=\alpha+\beta$, and similarly $\gamma'=\alpha+\gamma$.

Find a nonsingular $(s-\alpha)\times(s-\alpha)$ matrix U_2 and a nonsingular $t\times t$ matrix V_2 which together bring B_2 into Smith normal form, i.e.,

$$U_2B_2V_2=\begin{bmatrix} I_\beta & 0 \\ 0 & 0 \end{bmatrix}.$$

Then $M_2(X)$ has the same rank and deficiency as

$$M_3(X)=\begin{bmatrix} I & 0 & 0 \\ 0 & U_2 & 0 \\ 0 & 0 & I \end{bmatrix} M_2(X) \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & V_2 \end{bmatrix}$$

$$=\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & U_2B_2V_2 \\ 0 & C_2 & (X-C_1B_1)V_2 \end{bmatrix}.$$

Next find a nonsingular $t\times t$ matrix U_3 and a nonsingular $(s-\alpha)\times(s-\alpha)$ matrix V_3 which together bring C_2 into Smith normal form, i.e.,

$$U_3C_2V_3=\begin{bmatrix} I_\gamma & 0 \\ 0 & 0 \end{bmatrix}.$$

Then $M_3(X)$ has the same rank and deficiency as

$$M_4(X)=\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & U_3 \end{bmatrix} M_3(X) \begin{bmatrix} I & 0 & 0 \\ 0 & V_3 & 0 \\ 0 & 0 & I \end{bmatrix}$$

$$=\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & U_2B_2V_2 \\ 0 & U_3C_2V_3 & U_3(X-C_1B_1)V_2 \end{bmatrix},$$

which we repartition as

$$M_4(X)=\begin{bmatrix} I_\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_\beta & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & I_\gamma & 0 & Y_1 & Y_2 \\ 0 & 0 & 0 & Y_3 & Y_4 \end{bmatrix}, \tag{3}$$

where $$U_3(X-C_1B_1)V_2=\begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix}.$$

At this point β and γ are known. Assume without loss of generality that $\beta\leqslant\gamma$. Choose $Y_1=0$, $Y_2=0$, $Y_3=0$, and choose the $(t-\gamma)\times(t-\beta)$ matrix $Y_4=[I_{t-\gamma}0]$. On the one hand, this choice yields

$$\rho(M(X))=\rho(M_4(X))=\alpha+\gamma+\beta+(t-\gamma)$$
$$=\alpha+t+\min(\beta,\gamma)=t+\min(\beta',\gamma',),$$

so that equality holds in (1). On the other hand the choice is achieved by setting

$$X=C_1B_1+U_3^{-1}\begin{bmatrix} 0 & 0 & 0 \\ 0 & I_{t-\gamma} & 0 \end{bmatrix} V_2^{-1}. \tag{4}$$

If $\gamma<\beta$, then (4) can be replaced by

$$X=C_1B_1+U_3^{-1}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & I_{t-\beta} \end{bmatrix} V_2^{-1}; \tag{5}$$

i.e., $Y_1=0$, $Y_2=0$, $Y_3=0$, $Y_4^T=[0\ I_{t-\beta}]$.

The main computational labor involved comes (a) in bringing A into Smith normal form, (b) in determining V_2^{-1} and β, and (c) in determining U_3^{-1} and γ. Step (a) can be carried out by performing elementary row operations to "sweep out" A to upper triangular form (the product of these operators' matrices is U), and then performing elementary column operations (the product of whose matrices is V) to "sweep out" the resulting matrix into diagonal form $D_\alpha+0_{s-\alpha}$; it is not necessary to normalize D_α to I_α. As for step (b), we can take V_2 as the product of the matrices of elementary column operations used to sweep out B_2 into lower triangular form; each elementary matrix is trivially invertible so that V_2^{-1} can be built up during the sweep-out process, and β can be read off at the end of the process. Similarly for step (c).

(Paper 70B3–183)